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ON MOTION STABILITY RELATIVE TO A PART OF THE VARIABLES UNDER PERSISTENT PERTURBATIONS*

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The problem of stability and asymptotic stability of motion relative to a part of the variables under persistent perturbations is examined for the case when some of the latter may not be sufficiently small. Stability theorems of such kind are proved. A unified method based on a nonlinear change of variables and on differential inequalities is used to derive stability conditions for the motion of a solid body with one fixed point under persistent perturbations.

It is well known that the problem of motion stability relative to a part of the variables (y-stability) for linear systems is equivalent to the problem of Liapunov stability of motion for a certain auxiliary linear system whose dimension can be less than that of the original system. In the present paper a connection is established between the coefficients of the auxiliary system's characteristic equation and the coefficients of the original linear system. This permits a formulation of an algebraic criterion for the asymptotic y-stability of linear stationary systems, of algebraic conditions for complete controllability with respect to a part of the variables of a linear stationary controlled system, as well as of an analog of Popov's criterion yielding conditions of absolute y-stability of the motion of nonlinear controllable systems.

1. Let there be a linear stationary system of ordinary differential equations of perturbed motion

$$x^\cdot = Ax;\quad x = (y_1, \ldots, y_m,\ z_1, \ldots, z_p) = (y,\ z),\quad m > 0,\quad p > 0,\quad n = m + p$$

or, in the variables y, z

$$y^\cdot = Ay + Bz,\quad z^\cdot = Cy + Dz \tag{1.1}$$

where A, B, C, D are constant matrices of appropriate dimensions. Together with system (1.1) we consider the "perturbed" system

$$y^\cdot = Ay + Bz + R_y(t,\ y,\ z),\quad z^\cdot = Cy + Dz + R_z(t,\ y,\ z) \tag{1.2}$$

where the vector-valued functions R_y, R_z are persistent perturbations that are such that system (1.2) has a solution corresponding to each collection of initial data x_0, t_0. The components comprising the vector z and the vector-valued function R_z we divide into two groups and we represent z and R_z as $z = (z^+, z^-)$, $R_z = (R_z^+, R_z^-)$.

Definition 1. The motion $x = 0$ of system (1.1) is called $y(z^-)$-stable if for any number $\varepsilon > 0$ we can find positive numbers $\delta_i(\varepsilon)$ $(i = 1, 2)$, such that the inequality

$$t \geqslant 0,\ \|y(t;\ t_0,\ x_0)\| < \varepsilon,\ 0 \leqslant \|z(t;\ t_0,\ x_0)\| < +\infty \tag{1.3}$$

is fulfilled on all motions of system (1.2) starting in domain

$$\|y_0\| < \delta_1(\varepsilon),\quad \|z_0^+\| < \delta_1(\varepsilon),\ 0 \leqslant \|z_0^-\| < +\infty \tag{1.4}$$

for any values $R(t,\ y,\ z)$ satisfying the conditions

$$\|R_y(t,\ y,\ z)\| < \delta_2(\varepsilon),\quad \|R_z^+(t,\ y,\ z)\| < \delta_2(\varepsilon),\quad 0 \leqslant \|R_z^-(t, y, z)\| < +\infty \tag{1.5}$$

in domain (1.3). If, in addition $\lim \|y(t;\ t_0, x_0)\| = 0$, $t \to \infty$, then the motion $x = 0$ of system (1.1) is called asymptotically $y(z^-)$-stable.

Notes. 1°. If the vector z^- in (1.4) and the vector-valued function R^-_z in (1.5), respectively, the conditions $\|z_0^-\| < \delta_1(\varepsilon)$ and $\|R_z^-(t, y, z)\| < \delta_2(\varepsilon)$, then we shall say that the

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*Prikl.Matem.Mekhan.,Vol.47,No.2,pp.291-301,1983 motion $x=0$ of system (1.1) is $y(0)$-stable. When $m=n$ the definition of $y(0)$-stability leads to the well-known definition of stability under persistent perturbations /1/, and when $R_y\equiv 0,\ R_z\equiv 0$ to the definition of y-stability /2/.

2°. The definition of $y(z^-)$-stability makes sense only when $m<n$. Indeed, the presence in the system of perturbing factors arbitrary in magnitude leads to system (1.2) having equilibrium positions arbitrary in magnitude and, consequently, the problem of $x(z^-)$-stability makes no sense.

3°. The definition of asymptotic $y(z^-)$-stability and even of asymptotic $y(0)$-stability makes sense only when $m<n$ according to /1/.

Consider the matrices

$$K_p = (B^T,\ D^T,\ B^T,\ \ldots,\ D^{Tp-1}B^T) \tag{1.6}$$

$$L=\begin{Vmatrix} L_1 \\ L_2 \end{Vmatrix},\quad L_1=\begin{Vmatrix} E_m & 0 \\ & l_{11}\ldots l_{1p} \\ 0 & l_{h1}\ldots l_{hp} \end{Vmatrix} \tag{1.7}$$

where E_m is the unit $m\times m$-matrix, of size $(l_{i1},\ldots,l_{ip})^T,\ i=1,\ldots,h$ are linearly independent column-vectors of matrix K_p, L_2 is an arbitrary $(n-m-h)\times n$-matrix such that the matrix L is nonsingular, $h=\operatorname{rank} K_p$; T is the sign of transposition.

Theorem 1. Let the motion $x=0$ of system (1.1) be asymptotically y-stable. If in matrix K_p the rows numbered $i_1,\ldots,i_N$ are zero, then this motion is $y(z^-)$-stable and the variables z_s and the functions R_{zs} numbered $s=i_1,\ldots,i_N$, respectively, occur in the vector z^- and in the vector-valued function R_z^-

Proof. In system (1.1) we make the change of variables $\xi=Lx$, where matrix L is of form (1.7). In the new variables the equations of system (1.1), according to /3,4/, fall into two groups:

$$w^{\cdot}=A_1w,\quad v^{\cdot}=A_2w+A_3v,\quad \xi=(w,v)$$

and the $(m+h)$-dimensional vector w describing the state of the system

$$w^{\cdot}=A_1w \tag{1.8}$$

completely determines the behavior of the variables $y=(y_1,\ldots,y_m)$ of system (1.1). Together with (1.8) let us consider the system

$$w^{\cdot}=A_1w+L_1R,\quad R=(R_y,R_z)$$

According to /3/ the motion $w=0$ of system (1.8) is asymptotically Liapunov stable; therefore /1/, it is stable with respect to all variables under the persistent small perturbations L_1R. But the function L_1R does not contain the perturbations $R_{zs},\ s=i_1,\ldots,i_N$; therefore, the motion $x=0$ of system (1.1) is $y(z^-)$-stable, and the variables z_s and the functions R_{zs} numbered $s=i_1,\ldots,i_N$, respectively, occur in the vector z^- and in the vector-valued function R_z^-. The theorem is proved.

Corollary. If the motion $x=0$ of system (1.1) is asymptotically y-stable, then it is $y(0)$-stable.

Example 1. Let the Eqs.(1.1) of perturbed motion be

$$\begin{aligned} &y_1^{\cdot}=-y_1+z_1-2z_2 \\ &z_1^{\cdot}=4y_1+z_1+2z_2,\quad z_2^{\cdot}=8y_1+2z_1+4z_2 \end{aligned} \tag{1.9}$$

$$z_3^{\cdot}=2y_1+z_1+z_2-z_3$$

$$B=(1,\ 0,\ -2)$$

$$D=\begin{Vmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & -1 \end{Vmatrix},\quad \operatorname{rank} K_3=1,\quad K_1=\begin{Vmatrix} 1 \\ 0 \\ -2 \end{Vmatrix}$$

In this case the equations

$$w^{\cdot}=A_1w,\quad A_1=\begin{Vmatrix} -1 & 1 \\ 0 & -1 \end{Vmatrix}$$

comprise system (1.8). The eigenvalues of matrix A_1 have negative real parts and the second row of matrix K_1 is zero; therefore, according to Theorem 1, the motion $y_1=z_1=z_2=z_3=0$ of system (1.9) is $y_1(z^-)$-stable, and $z^-=z_2,\ R^-_z=R_{z_2}$. Thus, the unperturbed motion of system (1.9) is y_1-stable for any perturbing function R_{z_2} acting on the third equation and for perturbing functions $R_{y_1}, R_{z_1}, R_{z_3}$, sufficiently small in magnitude, acting on the other three equations of this system.

Example 2. Let us consider the equations of perturbed motion of a controllable system in the critical case of two zero roots

$$y_i^{\cdot} = \sum_{k=1}^{m} a_{ik} y_k + h_i f(\sigma), \quad i = 1, \ldots, m \tag{1.10}$$
$$z_1^{\cdot} = \gamma_1 f(\sigma), \quad z_2^{\cdot} = \gamma_2 f(\sigma)$$
$$\sigma = \sum_{k=1}^{m} \alpha_k y_k + \beta_1 z_1 + \beta_2 z_2 + \gamma_0$$

where $a_{ik}, h_i, \alpha_k, \gamma_1, \gamma_2, \beta_1, \beta_2, \gamma_0$ are constants, $f(\sigma)$ is a continuous function satisfying the condition $\sigma f(\sigma) > 0$, $\sigma \neq 0$. We introduce the new variables /5/ $\gamma\mu_1 = \beta_1 z_1 + \beta_2 z_2$, where $\gamma < 0$ is a constant number. System (1.10) reduces to

$$y_i^{\cdot} = \sum_{k=1}^{m} a_{ik} y_k + h_i f(\sigma), \quad i = 1, \ldots, m \tag{1.11}$$
$$\mu_1^{\cdot} = \Gamma f(\sigma), \quad \sigma = \sum_{k=1}^{m} \alpha_k y_k + \gamma\mu_1, \quad \Gamma = \frac{1}{\gamma}(\beta_1\gamma_1 + \beta_2\gamma_2)$$

The well-known conditions for the global stability of the unperturbed motion of system (1.11) /6/ will be, according to /5/, sufficient conditions for the global y-stability of the unperturbed motion of system (1.10) for any finite number γ_0, because the quantity γ_0/γ can be made sufficiently small by making a suitable choice of the quantitiy γ.

2. Let the vector-valued functions R_y and R_z in system (1.2) be

$$R_y = R_{y0} + R_y^*(t, y, z), \quad R_z = R_{z0} + R_z^*(t, y, z), \quad R = (R_y, R_z), \quad R^* = (R_y^*, R_z^*).$$

where R_{y0} and R_{z0} are constant vectors of appropriate dimensions. We assume that rank $K_p = h$ and by l_s $(s = 1, \ldots, h)$ we denote linearly-independent column-vectors of matrix K_p. Without loss of generality we shall take it that all column-vectors of matrix B^T are linearly independent. We consider the system of algebraic equations for determining λ_{ij} $(i, j = 1, \ldots, h)$

$$D^T l_i = \sum_{j=1}^{h} \lambda_{ij} l_j, \quad i = 1, \ldots, h$$

We assume that

$$l_j^T R_{z0} = \sum_{k=1}^{m} \lambda_{jk} R_{y0k} + \sum_{l=m+1}^{h} \lambda_{jl} R_{z0l}, \quad j = 1, \ldots, h \tag{2.1}$$

$$|R_i^*(t, y, z)| \leqslant \sum_{k=1}^{m} \alpha_{ik} |y_{ik}|, \quad i = 1, \ldots, n \tag{2.2}$$

where α_{ik} are sufficiently small positive constants.

Theorem 2. If the motion $x = 0$ of system (1.1) is asymptotically y-stable, then this motion will be asymptotically $y(0)$-stable under any sufficiently small perturbations R_{y0}, $l^T{}_j R_{z0}$ $(j = 1, \ldots, h)$ satisfying conditions (2.1) and any perturbations $R^*(t, y, z)$ satisfying conditions (2.2). If, additionally, the rows numbered $i_1, \ldots, i_N$ in matrix K_p are zero, then this motion is asymptotically $y(z^-)$-stable, and the variables z_s and the functions R_{zs} with numbers $s = i_1, \ldots, i_N$ occur in, respectively, the vector z^- and the vector-valued function R_z^-.

Proof. In view of condition (2.1), after the introduction of the new variables

$$\mu_i = l_i^T z^T + R_{y0i}, \quad i = 1, \ldots, m \tag{2.3}$$
$$\mu_{m+j} = l_{m+j}^T z^T + l_{m+j}^T R_{z0}, \quad j = 1, \ldots, h$$

the system

$$y^{\cdot} = Ay + Bz + R_{y0}, \quad z^{\cdot} = Cy + Dz + R_{z0} \tag{2.4}$$

reduces to

$$\eta^{\cdot} = A_1 \eta \tag{2.5}$$

where $\eta = (y, \mu)$ and μ is a h-dimensional vector consisting of the variables (2.3). The reduction of system (2.4) to form (2.5) is similar to the way in which in /3,4/ with $R_0 = 0$ the system (1.1) was reduced to a system of μ-form (1.8). The eigenvalues of matrix A_1 have negative real parts and, consequently, the motion $\eta=0$ of system (2.5) are asymptotically Liapunov-stable.

According to /7/, when (2.2) is fulfilled the motion $\eta=0, z=0$ of the nonlinear system

$$\eta^{\cdot} = A_1\eta + L_1R^*(t, x), \quad z^{\cdot} = Cy + Dz + R_z^*(t, x)$$

is asymptotically η-stable. Consequently, for any $\varepsilon > 0$, $t_0 \geqslant 0$ we can find $\lambda(\varepsilon, t_0) > 0$ such that from $\|\eta_0\| < \lambda$, $\|z_0\| < \lambda$ follows $\|\eta(t; t_0, \eta_0, z_0)\| < \varepsilon$ for all $t \geqslant t_0$ and, in addition, $\lim \|\eta(t; t_0, \eta_0, z_0)\| = 0$ as $t \to \infty$. With respect to λ and t_0 we can choose $\delta_i(\lambda, t_0) = \delta_i(\varepsilon, t_0) > 0$ $(i = 1, 2)$ such that from $\|x_0\| < \delta_1$, $\|R_{y0}\| < \delta_2$, $\|l_j^T R_{z0}\| < \delta_2$ $(j = 1, \ldots, h)$ follows $\|\eta(t; t_0, x_0)\| < \lambda$. Then for all $t \geqslant t_0$ we have $\|y(t; t_0, x_0)\| < \varepsilon$ and, in addition, $\lim \|y(t; t_0, x_0)\| = 0$ as $t \to \infty$. The theorem has been proved.

Example 3. Let the equations of perturbed motion (1.1) be of form

$$y_1^{\cdot} = -y_1 + z_1 - 2z_2, \quad z_1^{\cdot} = 4y_1 + z_1, \quad z_2^{\cdot} = 2y_1 + z_1 - z_2 \qquad (2.6)$$

$$l_1 = (1, -2)^T, \quad D^T = \begin{Vmatrix} 1 & 0 \\ 1 & -1 \end{Vmatrix}$$

Since $D^T l_1 = -l_1$, condition (2.1) in the case given becomes

$$R_{z_10} - 2R_{z_20} = -R_{y_10} \qquad (2.7)$$

After the introduction of the new variable $\mu_1 = z_1 - 2z_2 + R_{y_10}$ the system

$$y_1^{\cdot} = -y_1 + z_1 - 2z_2 + R_{y_10}, \quad z_1^{\cdot} = 4y_1 + z_1 + R_{z_10}$$
$$z_2^{\cdot} = 2y_1 + z_1 - z_2 + R_{z_20}$$

becomes

$$y_1^{\cdot} = -y_1 + \mu_1, \quad \mu_1^{\cdot} = -\mu_1$$

Therefore, with the fulfillment of conditions (2.2) the unperturbed motion of system (2.6) is asymptotically $y_1(0)$-stable in accord with Theorem 2.1.

3. We consider the motion of a heavy body with one fixed point, due to initial and to persistent perturbations. The equations of perturbed motion are

$$Ax_1^{\cdot} = (B - C)x_2x_3 + mg(x_{30}\gamma_2 - x_{20}\gamma_3) + A\Phi_1(t, x_1, x_2, x_3) \qquad (3.1)$$
$$\gamma_1^{\cdot} = x_3\gamma_2 - x_2\gamma_3 \ (123, ABC)$$

where A, B, C are the body's principle moments of inertia, $x_i (i=1, 2, 3)$ are the projections of the body's angular velocity onto the principal axes of inertia, $\gamma_i (i = 1, 2, 3)$ are the projections onto the principal axes of inertia of the unit vector directed along the fixed vertical axis, $x_{i0} (i = 1, 2, 3)$ are the coordinates of the body's center of inertia in the principle axes of inertia, $\Phi_i(t, x_1, x_2, x_3)$ $(i = 1, 2, 3)$ are the continuous persistent perturbations, $\Phi_i(t, 0, 0, 0) \equiv 0$ $(i = 1, 2, 3)$. We shall study the stability of the unperturbed motion of system (3.1) under a number of assumptions on the form of the functions $\Phi_i (i = 1, 2, 3)$.

1°. $\Phi_i(t, x_1, x_2, x_3) = \alpha_i x_i$, $\alpha_i = \text{const}$, $x_{i0} (i = 1,2,3)$, i.e., system (3.1) has the form

$$Ax_1^{\cdot} = \alpha_1 x_1 + (B - C)x_2x_3 (123, ABC) \qquad (3.2)$$

We introduce the new variable $\mu_1 = (B - C)x_2x_3/A$. Under the condition $C < A < B$ or $C > A > B$ we have the following estimations for system (3.2):

a) $x_1^{\cdot} = \alpha_1 x_1 + \mu_1$

$$\mu_1^{\cdot} = (\alpha_2 + \alpha_3)\mu_1 + x_1 \frac{B-C}{A}\left[\frac{C-A}{B}x_3^2 + \frac{A-B}{C}x_2^2\right] \leqslant (\alpha_2 + \alpha_3)\mu_1$$

in the domain

$$0 \leqslant x_1 \leqslant H, \quad 0 < |x_i| < +\infty \quad (i = 2, 3) \qquad (3.3)$$

b) $x_1^{\cdot} = \alpha_1 x_1 + \mu_1$, $\mu_1^{\cdot} \geqslant (\alpha_2 + \alpha_3)\mu_1$ in the domain

$$-H \leqslant x_1 \leqslant 0, \quad 0 \leqslant |x_i| < +\infty \quad (i = 2, 3) \qquad (3.4)$$

From the estimations a) and b) it follows that the variable $x_1(t)$ in system (3.2) is described by the equation

$$x_1^{\cdot} = \alpha_1 x_1 + \varphi(t), \quad |\varphi(t)| \leqslant x_2(t_0)x_3(t_0)\exp(\alpha_2 + \alpha_3)t$$

therefore, under the condition $\alpha_1 < 0$, $\alpha_2 + \alpha_3 < 0$ the motion $x_1 = x_2 = x_3 = 0$ of system (3.2) is asymptotically globally x_1-stable. If $\alpha_1 = 0$, $\alpha_2 + \alpha_3 < 0$ or $\alpha_1 < 0$, $\alpha_2 + \alpha_3 = 0$, then from estimations a) and b) follows the x_1-stability of the motion $x_1 = x_2 = x_3 = 0$.

Theorem 3. Let one of the three conditions

$$C < A < B, \quad B < A < C, \quad A = B \neq C \tag{3.5}$$

be fulfilled. If $\alpha_1 < 0$, $\alpha_2 + \alpha_3 < 0$, then the motion $x_1 = x_2 = x_3 = 0$ of system (3.2) is globally asymptotically x_1-stable. If $\alpha_1 < 0, \alpha_2 + \alpha_3 = 0$ or $\alpha_1 = 0, \alpha_2 + \alpha_3 < 0$, then this motion is (nonasymptotically) x_1-stable.

2°. $\Phi_i(t, x_1, x_2, x_3) = \alpha_i(t)x_i$, $\alpha_i(t)$ is piecewise continuous functions t, $x_{i0} = 0$ $(i = 1, 2, 3)$. System (3.1) has the form

$$Ax_1^{\cdot} = \alpha_1(t)x_1 + (B - C)\, x_2x_3 \quad (123,\ ABC) \tag{3.6}$$

Under the condition $C < A < B$ or $C > A > B$, for system (3.6) we have the estimations

$$x_1^{\cdot} = \alpha_1(t)x_1 + \mu_1, \quad \mu_1^{\cdot} \leqslant [\alpha_2(t) + \alpha_3(t)]\mu_1 \text{ in domain } (3.3)$$

$$x_1^{\cdot} = \alpha_1(t)x_1 + \mu_1, \quad \mu_1^{\cdot} \geqslant [\alpha_2(t) + \alpha_3(t)]\mu_1 \text{ in domain } (3.4)$$

Therefore, the variable $x_1(t)$ in system (3.6) is described by the equation

$$x_1^{\cdot} = \Gamma_1(t)\, x_1 + \varphi_1(t), \quad |\varphi_1(t)| \leqslant x_2(t_0)\, x_3(t_0) \exp\int_{t_0}^{t} \Gamma_2(\tau)\, d\tau$$

$$\Gamma_1(t) = \alpha_1(t), \; \Gamma_2(\tau) = \alpha_2(\tau) + \alpha_3(\tau)$$

and, consequently, the inequality

$$|x_1(t)| \leqslant |x_1(t_0)| \left\{\exp\int_{t_0}^{t} \Gamma_1(s)\, ds\right\} + \int_{t_0}^{t} \exp\left\{\int_{s}^{t} \Gamma_1(\tau)\, d\tau\right\}\left\{x_2(t_0)\, x_3(t_0) \int_{t_0}^{s} \Gamma_2(\theta)\, d\theta\right\} ds \tag{3.7}$$

is fulfilled.

Theorem 4. Let one of the three conditions (3.5) be fulfilled. If

$$\int_{t_0}^{t} \Gamma_i(\tau)\, d\tau < A_i, \quad A_i = \text{const} \quad (i = 1, 2)$$

$$\int_{t_0} \Gamma_i(\tau)\, d\tau \to -\infty, \quad t \to \infty$$

then the motion $x_1 = x_2 = x_3 = 0$ of system (3.6) is globally asymptotically x_1-stable.

The proof follows from inequality (3.7).

3°. $\Phi_1(t, x_1, x_2, x_3) = f_1(x_1)$, where $f_1(x_1)$ is a continuous function in the domain $|x_1| \leqslant H$; $\Phi_i(t, x_1, x_2, x_3) = \alpha_i x_i$, $\alpha_i = \text{const}$ $(i = 2, 3)$; $x_{i0} = 0$ $(i = 1, 2, 3)$. System (3.1) has the form

$$x_1^{\cdot} = f_1(x_1) + \frac{B - C}{A} x_2x_3, \quad x_2^{\cdot} = \alpha_2x_2 + \frac{C - A}{B} x_1x_3, \tag{3.8}$$

$$x_3^{\cdot} = \alpha_3x_3 + \frac{A - B}{C} x_1x_2$$

Under condition $C < A < B$ or $C > A > B$ we have the estimates

$$x_1^{\cdot} = f_1(x_1) + \mu_1, \quad \mu_1^{\cdot} = (\alpha_2 + \alpha_3)\, \mu_1 \text{ in domain } (3.3) \tag{3.9}$$

$$x_1^{\cdot} = f_1(x_1) + \mu_1, \quad \mu_1^{\cdot} = (\alpha_2 + \alpha_3)\, \mu_1 \text{ in domain } (3.4)$$

for system (3.8). Let us consider the system

$$\xi_1^{\cdot} = f_1(\xi_1) + \xi_2, \quad \xi_2^{\cdot} = (\alpha_2 + \alpha_3)\, \xi_2 \tag{3.10}$$

which is the comparison system for (3.9).

Theorem 5. Let one of the following two conditions be fulfilled: $C < A < B$ or $C > A > B$. If $(\alpha_2 + \alpha_3) f_1(\xi_1)\xi_1 > 0$, $f_1(\xi_1)/\xi_1 + (\alpha_2 + \alpha_3) < 0$ $(\xi_1 \neq 0)$

$$\int_0^{\xi_1} (\alpha_2 + \alpha_3) f_1(\xi_1)\, d\xi_1 \to \infty, \quad |\xi_1| \to \infty$$

then the motion $x_1 = x_2 = x_3 = 0$ of system (3.8) is globally asymptotically x_1-stable.

Proof. Under the fulfillment of the theorem's conditions the motion $\xi_1 = \xi_2 = 0$ of system (3.10) is globally asymptotically Liapunov-stable /8/; therefore, according to /9,10/, the motion $x_1 = x_2 = x_3 = 0$ of system (3.8) is globally asymptotically x_1-stable. The theorem is proved.

We consider the case $\alpha_2 + \alpha_3 = 0$. Then the behavior of the variable $x_1(t)$ in system (3.8) is determined, in view of estimations (3.9), by the equation

$$x_1^{\cdot} = f_1(x_1) + \varphi_2(t), \quad |\varphi_2(t)| \leqslant x_2(t_0)\, x_3(t_0)$$

According to the theorem on stability of motion under persistent perturbations /1/, the question of the x_1-stability of the motion $x_1 = x_2 = x_3 = 0$ of system (3.8) reduces to the question of the asymptotic Liapunov-stability of the motion $\xi = 0$ of the system $\xi^{\cdot} = f_1(\xi)$.

4°. $\Phi_1(t, x_1, x_2, x_3) = f_1(x_1)$; $\Phi_i(t, x_1, x_2, x_3) = f_i(x_2, x_3)$ $(i = 2, 3)$; $x_{i0} = 0$; f_i $(i = 1, 2, 3)$ are functions in domain $|x_i| \leqslant H$ $(i = 1, 2, 3)$, continous in all variables. System (3.1) has the form

$$Ax_1^{\cdot} = f_1(x_1) + (B - C)x_2x_3 \quad (123\ ABC) \tag{3.11}$$

Under the condition $C < A < B$ or $C > A > B$ we have the estimates

$$x_1^{\cdot} = f_1(x_1) + \mu_1, \quad \mu_1^{\cdot} \leqslant x_2 f_3 + x_3 f_2 \text{ in domain (3.3)} \tag{3.12}$$
$$x_1^{\cdot} = f_1(x_1) + \mu_1, \quad \mu_1^{\cdot} \geqslant x_2 f_3 + x_3 f_2 \text{ in domain (3.4)}$$

for system (3.11). Assume that

$$x_2 f_3 + x_3 f_2 = \psi(\mu_1) \tag{3.13}$$

where $\psi(\mu_1)$ is a continuous function in domain $|\mu_1| \leqslant H$.

Theorem 6. If the motion $\xi_1 = \xi_2 = 0$ of system

$$\xi_1^{\cdot} = f_1(\xi_1) + \xi_2, \quad \xi_2^{\cdot} = \psi(\xi_2)$$

is globally asymptotically Liapunov-stable, then the motion $x_1 = x_2 = x_3 = 0$ of system (3.11) is globally asymptotically x_1-stable.

The proof follows from (3.12), (3.13) and the results in /9,10/.

5°. $\Phi_i(t, x_1, x_2, x_3) = \alpha_i x_i$, $\alpha_i = \text{const}$ $(i = 1, 2, 3)$; $x_{10} = x_{20} = 0$, $x_{30} \neq 0$, $A = B \neq C$.

Theorem 7. If conditions $\alpha_1 < 0$, $\alpha_3 < 0$, $\alpha_2 + \alpha_3 < 0$ are fulfilled, the motion $x_1 = x_2 = x_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$ of system (3.1) is $(x_1, x_3, \gamma_1, \gamma_2, \gamma_3)$ -stable.

Proof. Under the assumptions made the estimates

$$x_1^{\cdot} = \alpha_1 x_1 + \mu_1 + \dot{\varphi}_3(t)$$
$$\mu_1^{\cdot} \leqslant (\alpha_2 + \alpha_3)\mu_1 - \varphi_4(t) \text{ in domain (3.3)}$$
$$\mu_1^{\cdot} \geqslant (\alpha_2 + \alpha_3)\mu_1 - \varphi_4(t) \text{ in domain (3.4)}$$
$$\left(\varphi_3(t) = \frac{1}{A} mgx_{30}\gamma_2, \quad \varphi_4(t) = -\frac{A - C}{A^2} mgx_{30}\gamma_1 x_3\right)$$

are valid for system (3.1). In view of the presence of the first integral $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ for system (3.1), its unperturbed motion is $(\gamma_1, \gamma_2, \gamma_3)$-stable. Since $\alpha_3 < 0$, for any $\varepsilon > 0$, $t_0 \geqslant 0$ we can find $\delta(\varepsilon, t_0) > 0$ such that from $|x_i(t_0)| < \delta$, $|\gamma_i(t_0)| < \delta$ $(i = 1, 2, 3)$ follows $|\varphi_i(t)| < \varepsilon$ $(i = 3, 4)$ for all $t \geqslant t_0$. Consequently, the motion $\xi_1 = \xi_2 = 0$ of system

$$\xi_1^{\cdot} = \alpha_1\xi_1 + \xi_2, \quad \xi_2^{\cdot} = (\alpha_2 + \alpha_3)\xi_2$$

is stable under small constant perturbations $\varphi_i(t)$ $(i = 3, 4)$ and, according to /9,10/, the motion $x_1 = x_2 = x_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$ of system (3.1) is $(x_1, x_3, \gamma_1, \gamma_2, \gamma_3)$-stable. The theorem is proved.

In conclusion let us show that the x_1-stability of the unperturbed motion of system (3.1), proved in paragraphs 1°–5°, is a more general concept than the x_1-stability defined by Rumiantsev in /2/. Indeed, in paragraphs 1°–5° it was shown that for any number $\varepsilon > 0$ we can find a positive number $\delta(\varepsilon) > 0$ such that from

$$|x_1(t_0)| < \delta, \quad |x_2(t_0)\, x_3(t_0)| < \delta \tag{3.14}$$
$$x_0 = (x_1(t_0),\ x_2(t_0),\ x_3(t_0))$$

follows $|x_1(t; t_0, x_0)| < \varepsilon$ for all $t \geqslant t_0$. The second inequality in (3.14) is possible when

$$|x_2(t_0)| < \delta_1, \quad |x_3(t_0)| < \Delta$$

or when

$$|x_2(t_0)| < \Delta, \quad |x_3(t_0)| < \delta_1$$

where δ_1 is sufficiently small and Δ is some finite (not small) number. Consequently, the initial perturbations in the determination of the x_1-stability of the unperturbed motion of system (3.1) need not be sufficiently small, as was assumed in /2/.

4. Let us formulate algebraic criteria for the asymptotic y-stability of the motion $x = 0$ of system (1.1). We assume that rank $K_p = h$ and we consider a matrix Q_i $(i = 1, \ldots, 5)$ of the following form:

a) the rows of the size $h \times p$-matrix Q_1 are the linearly-independent column-vectors of matrix K_p;

b) the columns of the size $h \times h$-matrix Q_2 are the linearly-independent column-vectors of matrix Q_1 (let these columns of matrix Q_1 have the numbers $i_1, \ldots, i_h$);

c) the row numbered i_s $(s = 1, \ldots, h)$ of the size $(n - m) \times h$-matrix Q_3 in the row numbered s of matrix Q_2^{-1}, while the remaining rows of matrix Q_3 are zero;

d) $Q_4 = \begin{Vmatrix} E_m & 0 \\ 0 & Q_1 \end{Vmatrix}$, $\quad Q_5 = \begin{Vmatrix} E_m & 0 \\ 0 & Q_3 \end{Vmatrix}$

Q_2^{-1} is the matrix inverse to matrix Q_2; E_m is the unit size $m \times m$-matrix.

Theorem 4.1. For the asymptotic y-stability of the motion $x = 0$ of system (1.1) it is necessary and sufficient that all the roots of the equation

$$| Q_4 A^* Q_5 - \lambda E_{m+h} | = 0 \tag{4.1}$$

have negative real parts.

Proof. According to /3/, the problem of the y-stability of motion for (1.1) is equivalent to the Liapunov-stability problem for a certain auxiliary stationary linear system $\zeta^\cdot = G\zeta$ (we call it a system of μ-form) of dimension $m + h$. Here the elements g_{ij} of matrix G are the elements numbered $i, j = 1, \ldots, m + h$ of the matrix LA^*L^{-1} in which L is of form (1.7). We denote the elements of matrix L^{-1} by $\{l_{ij}^-\}$ $(i, j = 1, \ldots, n)$. Since the columns numbered $i_1, \ldots, i_h$ of matrix Q_1, and thus also the columns numbered $m + i_s$ $(s = 1, \ldots, h)$ of matrix L_1, are linearly independent, matrix L can be represented in the form

$$L = \begin{Vmatrix} E_m & 0 \\ 0 & Q_1 \\ 0 & L_3 \end{Vmatrix}, \quad L_1 = \begin{Vmatrix} E_m & 0 \\ 0 & Q_1 \end{Vmatrix}, \quad L_2 = \| 0 \; L_3 \|$$

and the columns numbered $i_1, \ldots, i_h$ in matrix L_3 can be taken to be zero, while the remaining elements of matrix L_3 can be chosen such that $| L | \neq 0$. We take into account that

$$l_{ij}^- = [(-1)^{i+j} L_{ji}] / | L | \quad (i, j = 1, \ldots, n)$$

where L_{ji} is a minor of the determinant $| L |$ of matrix L, obtained from $| L |$ by the deletion of the jth row and the ith column. In addition, permutations of the columns in a square matric can change only the sign of its determinant, i.e.,

$$| L | = \begin{vmatrix} E_m & 0 \\ 0 & Q_1 \\ 0 & L_3 \end{vmatrix} = \begin{vmatrix} E_m & 0 & 0 \\ 0 & Q_2 & L_4 \\ 0 & 0 & L_5 \end{vmatrix} (-1)^k$$

where k is the number of permutations made of the column-vectors in matrix L, and in matrices L_4 and L_5 there occur, respectively, only those columns of matrix Q_1 that are not contained in Q_2 and in the nonzero matrices L_3. We shall have

$$l_{ij}^- = \begin{cases} 1, & i = j \\ 0, & i \neq j; \quad i, j = 1, \ldots, m \end{cases}$$

$$l_{ij}^- = 0 \quad (i = 1, \ldots, m; \; j = m + 1, \ldots, m + h)$$

$$(i = m + i_s, \; s = 1, \ldots, h; \; j = 1, \ldots, m)$$

$$(i = m + i_s, \; s \neq 1, \ldots, h; \; j = 1, \ldots, m + h)$$

$$l_{m+i_s, m+k} = [(-1)^{s+k} Q_{2ks}] / | Q_2 | \quad (k, s = 1, \ldots, h)$$

(Q_{2ks} is the minor of determinant $| Q_2 |$ resulting from the deletion of the k-th row and the sth column in $| Q_2 |$. Therefore,

$$Q_5 = \| l_{ij}^- \| \quad (i = 1, \ldots, n; \; j = 1, \ldots, m + h)$$

and, consequently

$$Q_4 A^* Q_5 = \| g_{ij} \| \quad (i, j = 1, \ldots, m + h)$$

The theorem is proved.

Note. Equation (4.1) is the characteristic equation of the system of μ-form equations introduced in /3/. Therefore, in Theorem (4.1), in contrast to the result in /3/, we have established a direct algorithmic connnection between the form of the coefficients in system (1.1) and the conditions for its y-stability.

Example 4. Let system (1.1) be of form (2.6). In this case $m=1$ and $p=2$, while

$$\operatorname{rank}(B^T,\ D^TB^T) = \operatorname{rank}\begin{Vmatrix} 1 & -1 \\ -2 & 2 \end{Vmatrix} = 1$$

We set up the matrices Q_i $(i=1,\ldots,5)$

$$Q_1 = \| 1 \ -2 \|,\quad Q_2 = \| 1 \|,\quad Q_3 = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}$$

$$Q_4 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{Vmatrix},\quad Q_5 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{Vmatrix},\quad Q_4A^*Q_5 = \begin{Vmatrix} -1 & 1 \\ 0 & -1 \end{Vmatrix}$$

Equation (4.1) becomes

$$| Q_4A^*Q_5 - \lambda E_2 | = (\lambda+1)^2 = 0 \tag{4.2}$$

The roots of Eq.(4.2) are negative; therefore, the motion $y_1 = z_1 = z_2 = 0$ of system (2.6) is asymptotically y_1-stable.

5. Let us obtain sufficient conditions for the complete controllability with respect to a part of the variables (complete y-controllability /4,11/) for the linear controlled system

$$x^{\cdot} = A^*x + B^*u;\quad x = (y_1,\ldots,y_m, z_1,\ldots,z_p) = (y, z),\quad m>0,\ p>0,\ n=m+p \tag{5.1}$$

in which x is the system's of n-dimensional state vector; $u = (u_1, \ldots, u_r)$ is the r-dimensional control vector; A^*, B^* are constant matrices of appropriate dimensions.

Theorem 5.1. If

$$\operatorname{rank}(Q_4B^*,\ Q_4A^*Q_5Q_4B^*,\ \ldots,\ (Q_4A^*Q_5)^{m+h-1}Q_4B^*) = m+h \tag{5.2}$$

then system (5.1) is completely y-controllable.

Proof. In system (5.1) we make the change of variables $\zeta = L_1x$, $x=(y,z)$. According to Theorem (4.1) the behavior of the variables occurring in vector ζ, is described by the equation

$$\zeta^{\cdot} = Q_4A^*Q_5\zeta + Q_4B^*u \tag{5.3}$$

Under the fulfillment of (5.2) system (5.3) is completely controllable /11/ and, consequently, system (5.1) is completely y-controllable. The theorem is proved.

6. Let us obtain sufficient conditions for the absolute y-stability /5/ for the nonlinear controllable systems /6/

$$x^{\cdot} = A^*x + bf(\sigma),\quad \sigma = ex$$
$$x = (y_1,\ldots,y_m, z_1,\ldots,z_p) = (y,z),\ m>0$$
$$p>0,\ n=m+p$$

or, in variables y, z

$$y^{\cdot} = Ay + Bz + b_1f(\sigma),\quad z^{\cdot} = Cy + Dz + b_2f(\sigma),\quad \delta = e_1y + e_2z \tag{6.1}$$

where $f(\sigma)$ is an arbitrary and continuous function satisfying the condition $0 \leqslant \sigma f(\sigma) \leqslant k\delta^2$, $A, B, C, D, b_1, b_2, e_1, e_2$ are constant matrices and vectors of appropriate dimensions. We assume that

$$\operatorname{rank}(B^*,\ D^TB^*,\ \ldots,\ D^{Tp-1}B^*) = \operatorname{rank} K_p^* = h$$

where $B^* = \| B^T, e_2^T \|$, and by applying rules a)–e) from Sect.4, we construct matrices Q_i^* $(i = 1,\ldots,5)$. (Rules a)–e) from Sect.4 are fulfilled here with the sole difference that in a) we need to take the matrix K_p^* instead of K_p).

Theorem 6.1. Let a real number q (without loss of generality we can assume that $q \geqslant 0$) exist such that the inequality

$$\frac{1}{k} + \operatorname{Re}(1+qi\omega)W(i\omega) > 0$$

$$(W(i\omega) = (eQ_5^*)(Q_4^*A^*Q_5^* - i\omega E_{m+h})Q_4^*b)$$

is fulfilled for all $\omega \geqslant 0$. Then the motion $x = 0$ of system (6.1) is absolutely y-stable.

Proof. As in Theorem (4.1) we can show that the equations

$$\zeta^{\cdot} = Q_4A^*Q_5\zeta + Q_4^*bf(\sigma), \quad \sigma = eQ_5^*\zeta \tag{6.2}$$

are a system of μ-form equations /5/ for (6.1), while the function $W(i\omega)$ is the frequency characteristic of the linear part of system (6.2). According to Popov's criterion /12/ and to the result in /5/, the motion $x = 0$ of system (6.1) is absolutely y-stable. The theorem is proved.

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REFERENCES

1. MALKIN I.G., Theory of Stability of Motion. Moscow, NAUKA, 1966.
2. RUMIANTSEV V.V., On stability of motion with respect to a part of the variables. Vestn. Mosk. Gos. Univ., Matem., Mekhan., Astron., Fiz., Khim., No.4, 1957. (See also L.-N.Y. Academic Press, 1971).
3. VOROTNIKOV V.I. and PROKOP'EV V.P., On stability of motion of linear systems with respect to a part of variables. PMM. Vol.42, No.2, 1978.
4. VOROTNIKOV V.I., On complete controllability and stabilization of motion relative to a part of variables. Avtom. i Telemekh., No.3, 1982.
5. VOROTNIKOV V.I., On stability of motion relative to part of the variables for certain non-linear systems. PMM Vol.43, No.3, 1979.
6. LUR'E A.I., Certain Nonlinear Problems of Automatic Control Theory. Moscow, GOSTEKHIZDAT, 1951.
7. OZIRANER A.S., On asymptotic stability and instability relative to part of the variables. PMM Vol.37, No.4, 1973.
8. BARBASHIN E.A., Introduction to Stability Theory. Moscow, NAUKA, 1967.
9. VOROTNIKOV V.I., On stability and stabilization of motion with respect to a part of the variables. PMM Vol.46, No.6, 1982.
10. MATROSOV V.M., Principle of comparison with a vector-valued Liapunov function. IV. Differents. Uravnen. Vol.5., No.12, 1969.
11. KRASOVSKII N.N., Theory of Control of Motion. Moscow, NAUKA, 1968. (See also, in English, Stability of Motion, Stanford Univ. Press, Stanford, Cal. 1963).
12. POPOV V.M., Absolute stability of nonlinear automatic control systems. Avtom. i Telemekh., No.8, 1962.

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